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$\mathcal{N} = 1$ theories and a geometric master field

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Abstract: We study the large-N limit of the class of U(N) $\mathcal{N} = 1$ SUSY gauge theories with an adjoint scalar and a superpotential $W(\Phi)$. In each of the vacua of the quantum theory, the expectation values $\langle \mathrm{Tr}\Phi^p \rangle$ are determined by a master matrix Φ_0 with eigenvalue distribution $\rho_{GT}(\lambda)$. $\rho_{GT}(\lambda)$ is quite distinct from the eigenvalue distribution $\rho_{MM}(\lambda)$ of the corresponding large-N matrix model proposed by Dijkgraaf and Vafa. Nevertheless, it has a simple form on the auxiliary riemann surface of the matrix model. Thus the underlying geometry of the matrix model leads to a definite prescription for computing $\rho_{GT}(\lambda)$, knowing $\rho_{MM}(\lambda)$.

Keywords: 1/N Expansion, Matrix Models.

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Contents

1. Introduction

For a long while after 'tHooft pointed out the emergence of riemann surfaces in the large N expansion [1], the relationship between gauge theories and geometry remained only a tantalising picture. However, in recent years following Maldacena's conjecture [2], we have begun to understand more clearly the nature of the gauge theory/geometry correspondence.

Topological strings provide a very tractable context in which to precisely study this correspondence. By now, the original duality of this kind [3] between large N Chern-Simons theory and closed topological strings has been understood at different levels (See for e.g. [4]). Subsequently, embedding these topological string dualities in physical string theory [5] has led to a lot of new insights. For instance, Cachazo, Intriligator and Vafa [6] pointed out the geometric origin of the quantum superpotential of a large class of $\mathcal{N} = 1$ theories in this way. These results and the relation to topological strings led Dijkgraaf and Vafa [7] to the striking conjecture that these gauge theory superpotentials were determined by a large-N zero dimensional matrix model. This conjecture has been subsequently generalised and checked in various other cases [8]–[25]. Very recently a pertubative field theory argument for the localisation of these gauge theory superpotentials to zero dimensional matrix integrals, has also been given [9, 26].

This geometrising of our understanding of large-N gauge theories means that in some sense, the geometry is the master field [27, 28] of the gauge theory (as also observed in [9, 29]). Recall that the master field is the infinite N space-time independent field configuration whose classical expectation values reproduce the leading large-N quantum expectation values of all gauge invariant correlation functions.

$$F(A_0) = \langle F(A) \rangle \,, \tag{1.1}$$

where F is an arbitrary gauge invariant functional of the gauge field A and A_0 is the corresponding master field configuration. In the case of matrix models this master field configuration is nothing other than the large-N saddle point of the matrix integral [30].

The relation to the planar matrix model for $\mathcal{N}=1$ theories holds out hope of obtaining a simple characterisation of the master field of these theories (at least in the holomorphic sector). In fact, one might guess that the large-N saddle point of the matrix integral would play the role of the master field. This is not quite the case. Nevertheless, in this note we will make a beginning in trying to understand how exactly the matrix model (and its underlying geometry) can play the role of the master field, at least, for the class of $\mathcal{N}=1$ theories studied in [6].

These $\mathcal{N}=1$ $\mathrm{U}(N)$ gauge theories, have an adjoint chiral superfield Φ with a tree level superpotential $W(\Phi)$ which is generally taken to have a polynomial form

$$W(\Phi)=\sum_{p=2}^{n+1}\frac{g_p}{p}\mathrm{Tr}\Phi^{\mathrm{p}}\,. \tag{1.2}$$

There are many quantum vacua of the theory each labelled by a specific pattern of gauge symmetry breaking. For a given tree level $W(\Phi)$, Dijkgraaf and Vafa gave a prescription for obtaining the effective superpotential (even for the finite rank theory) from the eigenvalue density of the large-N matrix model with potential $W(\Phi)$ [7].

To make progress towards identifying a master field in the matrix model (or geometry) description, we will also need to look at other holomorphic quantities of interest in the gauge theory (GT for short). In fact, the class of observables that are natural to examine in this context are the expectation values, in each of these vacua, of $\langle Tr\Phi^p\rangle_{GT}$ (for arbitrary p, in the limit of large-N).[1] The solution of the gauge theory via deformation of the underlying $\mathcal{N}=2$ theory [6, 32] contains the answer, in principle, for the values of these observables.

Here, we will merely make the observation that this information is also naturally contained in the matrix model, being encoded in the associated riemann surface. However, belying naive expectations, $\langle Tr\Phi^p\rangle_{GT}$ are *not* the same as the expectation values $\langle Tr\Phi^p\rangle_{MM}$ in the matrix model.[2] Thus the gauge theory master field matrix Φ_0 (which would reproduce the gauge theory moments, $Tr\Phi_0^p=\langle Tr\Phi^p\rangle_{GT}$) is not the saddle point of the large-N matrix integral. Instead, Φ_0 is characterised by a distinct eigenvalue distribution $\rho_{GT}(\lambda)$. We can, however, give a definite prescription for extracting $\rho_{GT}(\lambda)$ from knowing the matrix eigenvalue distribution $\rho_{MM}(\lambda)$. We will see that despite being distinct, $\rho_{GT}(\lambda)$ and $\rho_{MM}(\lambda)$ have many similarities including the fact that they have identical support over the same intervals.

There are actually good reasons why we should not have expected $\langle Tr\Phi^p\rangle_{GT}$ to coincide with $\langle Tr\Phi^p\rangle_{MM}$. For a general (1.2) there is always a confining vacuum — where the gauge symmetry is unbroken. The low energy physics in this vacuum should be given by pure

[1]Note that the low energy superpotential $W_{low}(g_p,\Lambda)$ obtained from the Dijkgraaf-Vafa prescription gives us some information in this regard: it computes $\langle Tr\Phi^p\rangle$ for $p\leq n+1$. For instance, in the case of a cubic superpotential we have $\frac{\partial W_{low}}{\partial g_2}=\langle Tr\Phi^2\rangle$ and $\frac{\partial W_{low}}{\partial g_3}=\langle Tr\Phi^3\rangle$. Higher moments can be obtained through a deformation of the superpotential. See [31].

[2]This was also observed in the matrix model analysis of the confining vacuum of the $\mathcal{N}=1^*$ theory [11].

$\mathcal{N}=1$ super Yang-Mills and this should be reflected in $\langle Tr\Phi^p\rangle_{GT}$ (and thus $\rho_{GT}(\lambda)$) being independent of the detailed form of the superpotential $W(\Phi)$. But in the matrix model, $\rho_{MM}(\lambda)$ depends very strongly on the form of $W(\Phi)$. Thus, clearly the two cannot coincide in general. We will see explicitly how our prescription will nonetheless enable us to extract out the universal $\rho_{GT}(\lambda)$ out of the nonuniversal $\rho_{MM}(\lambda)$ for these cases.

We should note here that the master field Φ_0 is contructed to reproduce the holomorphic observables defined above. To compute more general nonholomorphic gauge invariant quantities, we will need master fields for the gauge field etc. This still remains an open problem.

In the next section we review the solution to the $\mathcal{N}=1$ gauge theory and the bare bones of the Dijkgraaf-Vafa prescription. In section 3, we proceed from this solution to show how the matrix model knows about the density of eigenvalues $\rho_{GT}(\lambda)$ of the gauge theory. We give a calculationally precise prescription for extracting this from the matrix model solution. In section 4 we illustrate this prescription with a couple of simple examples.

2. Review

In this section we review known results of [6, 32, 7] keeping more or less to the original notation of those papers.

The theory with a tree level superpotential (1.2) has classical vacua preserving $\mathcal{N}=1$ SUSY when the eigenvalues of Φ take values in the set of zeroes $\{a_i\}$ (assumed for simplicity to be distinct) of $W'(x)=g_{n+1}\prod_{i=1}^{n}(x-a_i)$. For a $\mathrm{U}(N)$ gauge theory, the vacua can then be labelled by the positive integers N_i corresponding to the number of eigenvalues taking the value a_i (with $\sum N_i=N$). Classically the gauge symmetry is broken

$$\mathrm{U}(N)\rightarrow\prod_{i=1}^{n}\mathrm{U}(N_i)\,. \tag{2.1}$$

Quantum mechanically, only the n $\mathrm{U}(1)$'s in the product gauge group survive in the low energy theory since the $\mathrm{SU}(N_i)$ sectors are confining. There is a low energy effective superpotential $W_{low}(g_p,\Lambda)$ corresponding to each of these vacua, where Λ is the dynamically generated scale of the underlying $\mathrm{SU}(N)$ theory. Sometimes, it is convenient to think in terms of a Veneziano-Yankielowicz type effective superpotential $W_{\text{eff}}(S_i,g_p,\Lambda_i)$ which depends on glueball superfields $S_i=Tr_{\mathrm{SU}(N_i)}W_\alpha W^\alpha$ and scales Λ_i. W_{low} is then obtained by minimising W_{eff} w.r.t. the S_i and then evaluating it on the minimum.

We can extract the information about holomorphic quantities in the quantum vacua (for instance, W_{low}) from the Seiberg-Witten (SW) curve [33]

$$\tilde{y}^2=P_N^2(x)-4\Lambda^{2N} \tag{2.2}$$

of the corresponding $\mathrm{U}(N)$ $\mathcal{N}=2$ Yang-Mills theory [34, 35]. This is because the vacua of the $\mathcal{N}=1$ theory are points on the coulomb branch of the $\mathcal{N}=2$ theory [33]. In the spirit of [36] Cachazo, Intriligator and Vafa [6] observed that the symmetry breaking in (2.1) implies that the $\mathcal{N}=1$ vacua would lie on submanifolds of the coulomb branch where the SW curve factorises as

$$\tilde{y}^2 = P_N^2(x) - 4\Lambda^{2N} = F_{2n}(x) H_{N-n}^2(x) \tag{2.3}$$

where F_{2n} and H_{N-n} represent polynomials of degree $2n$ and $N-n$ respectively. This factorisation with $N-n$ double roots reflects $N-n$ (mutually local) monopoles becoming massless. The $\mathcal{N}=1$ vacua are the points on this submanifold where the tree level superpotential would be minimised. Further, as was proven by Cachazo and Vafa [32], these minima are uniquely determined[3] by requiring that $F_{2n}(x)$ be of the form

$$g_{n+1}^2 F_{2n} = W'(x)^2 + f_{n-1}(x)\,, \tag{2.4}$$

i.e. it is a deformation of the classical ($\Lambda \to 0$) answer by a polynomial of degree $n-1$. The effect of this deformation is to split the double roots a_i of $W'(x)^2$ into (generically) distinct pairs of roots (a_i^-, a_i^+).

The crucial observation [6] (motivated by the large-N dual Calabi-Yau geometry) then is that the reduced riemann surface specified by

$$y^2 = F_{2n}(x) \tag{2.5}$$

completely determines W_{eff} and thus W_{low} (as also the U(1) gauge couplings). The genus $n-1$ hyperelliptic riemann surface in (2.5) has n branch cuts α_i between the n pairs of roots (a_i^-, a_i^+) of F_{2n}. Then W_{eff} is given by

$$S_i = \frac{g_{n+1}}{2\pi i}\oint_{\alpha_i} y\,dx\,; \qquad \Pi_i = \frac{g_{n+1}}{2\pi i}\int_{C_i} y\,dx\,; \qquad W_{\text{eff}} = \sum_i (N_i \Pi_i + \tau_{YM} S_i)\,, \tag{2.6}$$

where the curves C_i start from the ith branch cut and go off to infinity in the x plane (see figure1) of [32]. Unlike the original SW curve which has a strong dependence on N, the curve (2.5) is universal in the sense that it depends only on the pattern of symmetry breaking as specified by the ratios $\nu_i = \frac{N_i}{N}$.

More precisely, consider $\mathrm{U}(K) \to \prod_{i=1}^n \mathrm{U}(K_i)$ with $\sum_i K_i = K$, with the K_i having no common divisor. We are then localised to points on the coulomb branch where the SW curve satisfies

$$P_K^2 - 4\Lambda^{2K} = F_{2n} H_{K-n}^2(x) \tag{2.7}$$

with F_{2n} as in (2.4). Now consider the U(N) theory (with $N = MK$) with the same classical superpotential $W(\Phi)$ and the same pattern of symmetry breaking, namely, $\mathrm{U}(N) \to \prod_{i=1}^n \mathrm{U}(N_i)$, where $N_i = MK_i$. It is easy then to check that

$$P_{(N=MK)}(x) = \tilde{\Lambda}^{MK} T_M\left(\frac{P_K(x)}{\Lambda^k}\right) \tag{2.8}$$

satisfies

$$P_{MK}^2 - 4\tilde{\Lambda}^{2MK} = F_{2n} H_{MK-n}^2(x)\,, \tag{2.9}$$

[3] We also require that in the $\Lambda \to 0$ limit, $P_N(x) \to \prod_{i=1}^n (x-a_i)^{N_i}$.

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where

$$H_{MK-n}(x) = \frac{\tilde{\Lambda}^{MK}}{\Lambda^k} U_{M-1}\left(\frac{P_K(x)}{\Lambda^k}\right) H_{K-n}(x)\,. \tag{2.10}$$

Here $T_M(z = 2\cos\theta) = 2\cos M\theta$ and $U_{M-1}(z = 2\cos\theta) = \frac{\sin M\theta}{\sin\theta}$ are the usual Chebyshev polynomials.

The point here is that the SW curve of the U(MK) theory has the same factorisation as the U(K) theory with the *same* F_{2n}. Thus we associate the same riemann surface (2.5) to all these theories. This riemann surface can only depend on the fractions ν_i. In fact, this is why the geometry is able to capture even the finite K superpotential. But we can also take $M \to \infty$ and thus take the large-N limit while preserving the pattern of symmetry breaking. W_{eff}, and after minimisation W_{low}, are determined by (2.6) in terms of the riemann surface (2.5).

Dijkgraaf and Vafa [7] observed that (2.5) is nothing other than the riemann surface associated to the large-N limit of the zero dimensional hermitian matrix model

$$Z_{MM} = \exp\left(-\frac{N^2}{\mu^2}F_0\right) = \int [D\Phi] \exp\left(-\frac{N}{\mu}\,\mathrm{Tr}\,W(\Phi)\right), \tag{2.11}$$

expanded about the vacuum with a fractional pattern of symmetry breaking (not necessarily related to that of the gauge theory). This connection was also motivated by the fact that the superpotential of this gauge theory is captured by an open topological string theory which reduces to this matrix model.

The prescription in the matrix model to compute the superpotential is then as follows. From the solution of the matrix model, the saddle point eigenvalue distribution $\rho_{MM}(\lambda)$ for (2.11) can be seen to be proportional to the discontinuity of the meromorphic one form ydx across the branch cuts α_i. Therefore $\oint_{\alpha_i} ydx$ now has the matrix model interpretation of being proportional to the fraction of eigenvalues supported on the cut α_i. Comparing with (2.6) we need the identification

$$\frac{g_{n+1}}{2\pi i}\oint_{\alpha_i} ydx = S_i\,. \tag{2.12}$$

Given S_i this equation can be used to determine the coefficents of f_{n-1} that appears in F_{2n}. The planar free energy F_0 expanded about the vacuum of the matrix model with this distribution of eigenvalues then obeys

$$\frac{\partial F_0}{\partial S_i} = \frac{g_{n+1}}{2\pi i}\int_{C_i} ydx = \Pi_i\,. \tag{2.13}$$

We can then compute W_{eff} using the last relation in (2.6) and thus W_{low} after minimisation w.r.t. the S_i. Our interest, in what follows, will be in the particular riemann surface underlying the matrix model after S_i have taken their minimum values $\langle S_i\rangle$.

3. The master matrix Φ_0

We first obtain the master matrix Φ_0 in the gauge theory by putting together the various ingredients of the solution of the previous section. We will then see that it is given in terms of a particularly simple form on the riemann surface (2.5). Thus we will be able to give

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a prescription of how to obtain the eigenvalue distribution of Φ_0 from the matrix model. Though distinct from $\rho_{MM}(\lambda)$ it is nevertheless completely determined in terms of $\rho_{MM}(\lambda)$.

Our considerations will be for a fixed (but arbitrary) tree level $W(\Phi)$ as in (1.2). Let's start with a $\mathrm{U}(K)\to\prod_{i=1}^{n}\mathrm{U}(K_i)$ vacuum. Since by definition $P_K(x)=\langle det(x-\Phi)\rangle$, the K roots of P_K determine $\langle Tr\Phi^p\rangle_{GT}$ for $p\leq K$. We can study the properties of the vacuum (with the same pattern of symmetry breaking) in the large $N=MK$ limit by scaling all the K_i by a common factor M (as in the previous section) and taking $M\to\infty$.

Since, by (2.8), $P_{MK}(x)=\tilde{\Lambda}^{MK}T_M(\frac{P_K(x)}{\Lambda^k})$, the ($N=MK$) roots of $P_{MK}(x)$ are given by

$$P_K\left(\lambda_m^{(k)}\right)=2\Lambda^K\cos\frac{(2m+1)\pi}{2M}\,,\qquad(m=0,\ldots,M-1,k=1,\ldots,K)\,. \tag{3.1}$$

Here the right hand side comes from the M roots of the Chebyshev polynomial T_M. Taking $M\to\infty$, we see from the r.h.s that there is a uniform distribution on the semicircle $\theta\in[0,\pi]$. It is easy then to verify that the distribution of the roots λ_m^k is given by

$$\rho_{GT}(\lambda)\equiv\frac{1}{\pi}\frac{d\theta}{d\lambda}=\frac{1}{\pi K}\frac{dP_K(\lambda)}{d\lambda}\frac{1}{\sqrt{4\Lambda^{2K}-P_K^2(\lambda)}} \tag{3.2}$$

such that

$$\frac{1}{N}\langle Tr\Phi^p\rangle_{GT}=\int\lambda^p\rho_{GT}(\lambda)d\lambda\,. \tag{3.3}$$

Thus, from the solution to the gauge theory we see that the master matrix Φ_0 has an eigenvalue distribution $\rho_{GT}(\lambda)$ given by (3.2). This distribution has also appeared in [32] the context of the geometric dual to the $\mathcal{N}=2$ Seiberg-Witten theory.

We would now like to see if we can give a prescription to directly obtain this distribution from the matrix model and the associated reduced riemann surface (2.5). From the form (3.2) it is not obvious that we can do so. After all the matrix model does not know about K. It is only sensitive to the filling fractions $\nu_i=\frac{K_i}{K}$.

However, notice that,

$$P_K^2-4\Lambda^{2K}=F_{2n}H_{K-n}^2(x) \tag{3.4}$$

implies that

$$P_K\frac{dP_K(x)}{dx}=H_{K-n}(x)\left[2H'_{K-n}F_{2n}+H_{K-n}F'_{2n}\right]\equiv H_{K-n}Q_{K+n-1} \tag{3.5}$$

where $'$ denotes differentiation with respect to x. Since P_K has no roots in common with H_{K-n} (that would contradict (3.4)), (3.5) implies that $Q_{K+n-1}(x)$ has $P_K(x)$ as a factor

$$Q_{K+n-1}(x)=P_K(x)R_{n-1}^{(K)}(x)$$

where we have defined a degree $n-1$ polynomial $R_{n-1}^{(K)}(x)$ and the superscript is a reminder that it can still depend on K even if its degree does not. Therefore,

$$\frac{dP_K(x)}{dx}=H_{K-n}(x)R_{n-1}^{(K)}(x) \tag{3.6}$$

and hence (3.2) simplifies (using (3.4) and (3.6)) to

$$\rho_{GT}(\lambda) = \frac{1}{\pi K}\frac{dP_K(\lambda)}{d\lambda}\frac{1}{\sqrt{4\Lambda^{2K} - P_K^2(\lambda)}} = \frac{1}{\pi K}\frac{R_{n-1}^{(K)}(\lambda)}{\sqrt{-F_{2n}(\lambda)}}\,. \tag{3.7}$$

We will now see that

$$R_{n-1}(\lambda) = \frac{1}{K}R_{n-1}^{(K)}(\lambda)$$

is actually independent of the overall size K of the gauge group, i.e. $(1/K)R_{n-1}^{(K)}(\lambda) = \frac{1}{MK}R_{n-1}^{(MK)}(\lambda)$ for any M. Then $R_{n-1}(\lambda)$ would depend only on the pattern of symmetry breaking represented by the ratios ν_i.

This result follows from (2.8) which after differentiating gives

$$\begin{aligned}
\frac{dP_{MK}(x)}{dx} &= \frac{\tilde{\Lambda}^{MK}}{\Lambda^K}MU_{M-1}\left(\frac{P_K(\lambda)}{\Lambda^k}\right)\frac{dP_K(\lambda)}{d\lambda} \\
&= \frac{\tilde{\Lambda}^{MK}}{\Lambda^K}MU_{M-1}\left(\frac{P_K(\lambda)}{\Lambda^k}\right)H_{K-n}(\lambda)R_{n-1}^{(K)}(\lambda) \\
&= MH_{MK-n}(x)R_{n-1}^{(K)}(\lambda).
\end{aligned} \tag{3.8}$$

In obtaining the first line we have used the property of Chebyshev polynomials that $\frac{dT_M(x)}{dx} = MU_{M-1}(x)$. The next two lines follow from eqs. (3.6) and (2.10). Comparing the last line with eq. (3.6) for K replaced by MK, namely,

$$\frac{dP_{MK}(\lambda)}{d\lambda} = H_{MK-n}(\lambda)R_{n-1}^{(MK)}(\lambda)\,,$$

we see that

$$\frac{1}{MK}R_{n-1}^{(MK)}(\lambda) = \frac{1}{K}R_{n-1}^{(K)}(\lambda)\,. \tag{3.9}$$

Thus proving the claim of the previous paragraph.

As a result, we see that (3.7) can be written in the universal form

$$\rho_{GT}(\lambda) = \frac{1}{\pi}\frac{R_{n-1}(\lambda)}{\sqrt{-F_{2n}(\lambda)}}\,. \tag{3.10}$$

As mentioned in section 2, the eigenvalue value density $\rho_{MM}(\lambda)$ of the matrix model with potential $W(\Phi)$ expanded about a vacuum with filling fractions $\langle S_i\rangle$ is determined by the one form ydx as in (2.12) to be

$$\frac{\mu}{g_{n+1}}\rho_{MM}(\lambda) = \frac{1}{2\pi}\sqrt{-F_{2n}(\lambda)}. \tag{3.11}$$

Thus the two distributions are quite distinct, though we see that they have the same cut structure and thus identical support over the n branch cuts of the riemann surface (2.5). Classically, the eigenvalues of the matrix model sit at the extrema a_i of the potential $W(\Phi)$. As do the eigenvalues of Φ in the gauge theory. We see that quantum mechanically, the eigenvalues of both spread over the intervals $[a_i^-, a_i^+]$ defined by the branch cuts of $\sqrt{-F_{2n}}$. Thus the $\rho_{MM}(\lambda)$ and $\rho_{GT}(\lambda)$ are temptingly similar but nonetheless distinct. And for good reason, too, as we argued in the introduction.

Can we, knowing the large-N solution $\rho_{MM}(\lambda)$ to the matrix model, reconstruct $\rho_{GT}(\lambda)$? The answer is yes, essentially because $\rho_{GT}(\lambda)$ can be expressed in terms of a natural one form on the riemann surface determined by the matrix model solution. In fact, $\rho_{GT}(\lambda)$ can be written in terms of the meromorphic one form

$$\omega = \frac{R_{n-1}(x)dx}{y} \tag{3.12}$$

on the riemann surface $y^2 = F_{2n}(x)$. The discontinuity of ω across the cuts is proportional to $\rho_{GT}(\lambda)$. In fact,

$$\frac{1}{2\pi i}\oint_{\alpha_i}\omega = \int_{a_i^-}^{a_i^+}\rho_{GT}(\lambda)d\lambda = \nu_i\,. \tag{3.13}$$

The last equality follows from the fact that the fraction of gauge theory eigenvalues in the ith vacuum is ν_i. It can also be seen to follow from the form of $\rho_{GT}(\lambda)$ in (3.2). This fact will enable us to construct ω knowing ydx or equivalently $\rho_{MM}(\lambda)$.

Note that the $n-1$ one forms

$$\frac{x^k dx}{y}\,, \qquad (k = 0, \ldots, n-2)$$

are holomorphic on our riemann surface and form a basis for the space of holomorphic one forms (for this and other facts about hyperelliptic surfaces used below, see for example [37] or [38]). On the other hand, $\frac{x^{n-1}dx}{y}$ has a simple pole at each of the two preimages of $x=\infty$ (with residue ± 1). Thus ω is a one form which has only (two) simple poles coming from the highest power of x in $R_{n-1}(x)$. From the definition of R_{n-1} (see (3.6) and below) and since the coefficent of the highest power of x in P_K, as is that of F_{2n} (from eq. (2.4)), we conclude that the coefficent of x^{n-1} in $R_{n-1}(x)$ is also one. Therefore ω has residues ± 1 at its poles.

Note then that the one form

$$\omega' = \omega - \frac{x^{n-1}dx}{y}$$

is a holomorphic one form by construction. We know its A-periods, since we know the periods of ω and we can calculate the periods of $\frac{x^{n-1}dx}{y}$. Knowing the A-periods uniquely determines a holomorphic form — we have a unique expansion in the basis of holomorphic forms given above. We can thus construct ω' and therefore ω and $\rho_{GT}(\lambda)$ uniquely, once we know the riemann surface from the solution to the matrix model.

4. Examples

It might help if the general considerations of the previous section were illustrated with a couple of examples.

4.1 Gaussian model

The simplest case to study is a U(N) theory with quadratic superpotential

$$W(\Phi) = \frac{m}{2} Tr\Phi^2$$

There is only one vacuum, classically at $\Phi = 0$ and with unbroken gauge symmetry.

The corresponding large-N gaussian matrix model

$$Z = \int [D\Phi] \exp\left(-\frac{Nm}{2\mu} Tr\Phi^2\right). \tag{4.1}$$

has the well known Wigner semicircular distribution

$$\rho_{MM}(\lambda) = \frac{1}{2\pi\Lambda^2}\sqrt{4\Lambda^2 - \lambda^2}\,, \qquad \lambda \in [-2\Lambda, 2\Lambda]\,, \qquad \left(\Lambda^2 = \frac{\mu}{m}\right). \tag{4.2}$$

The planar moments are given by

$$\frac{1}{N}\langle Tr\Phi^{2k}\rangle_{MM} = \frac{(2k)!}{k!(k+1)!}\Lambda^{2k}\,. \tag{4.3}$$

Let's apply the prescription of the previous section to obtain $\rho_{GT}(\lambda)$. From (4.2) and (2.12) it follows that $y^2 = x^2 - 4\Lambda^2$. Since $n = 1$, we have from (3.12) and (3.13) that

$$\omega = \frac{dx}{y} \Rightarrow \rho_{GT}(\lambda) = \frac{1}{\pi}\frac{1}{\sqrt{4\Lambda^2 - \lambda^2}}\,, \qquad \lambda \in [-2\Lambda, 2\Lambda]\,. \tag{4.4}$$

$\rho_{GT}(\lambda)$ gives rise to the distinct moments

$$\frac{1}{N}\langle Tr\Phi^{2k}\rangle_{GT} = \frac{(2k)!}{(k!)^2}\Lambda^{2k}\,. \tag{4.5}$$

This distribution of eigenvalues (4.4) in the gauge theory goes back to the work of [36]. It arises from the factorisation of the SW curve in this vacuum as [36]

$$P_N(x) = \Lambda^N T_N\left(\frac{x}{\Lambda}\right) \Rightarrow P_N^2(x) - 4\Lambda^{2N} = U_{N-1}^2\left(\frac{x}{\Lambda}\right)\left(x^2 - 4\Lambda^2\right). \tag{4.6}$$

We see that in both the gauge theory and the matrix model, that the eigenvalues which were classically at zero are now spread over the interval $[-2\Lambda, 2\Lambda]$.

As an aside, we point out a curious connection of the two distributions in (4.2) and (4.4) with earlier work on the master field. One representation of the master field for the gaussian matrix model is in terms of the Cuntz oscillators [39, 40, 41, 42] a and $a^\dagger$ obeying $aa^\dagger = 1$, $a^\dagger a = 1 - |0\rangle\langle 0|$ with $a|0\rangle = 0$. Then

$$\hat{M} = \Lambda(a + a^\dagger) \Rightarrow \langle 0|\hat{M}^{2k}|0\rangle = \frac{1}{2\pi\Lambda^2}\int \lambda^{2k}\sqrt{4\Lambda^2 - \lambda^2}d\lambda\,. \tag{4.7}$$

In other words, $\hat{M}$ above is the master field for the gaussian model (see for example [41] for more details). What we would like to point out here is that

$$\mathrm{Tr}\,\hat{M}^{2k} = \frac{1}{\pi}\int \lambda^{2k}\frac{1}{\sqrt{4\Lambda^2 - \lambda^2}}d\lambda\,. \tag{4.8}$$

In other words, the operator trace (as opposed to cuntz vacuum expectation values) of powers of $\hat{M}$ reproduces the gauge theory moments and in this sense is the master field for the gauge theory as well. The significance of this is not clear. The fact that $\hat{M}$ has the same eigenvalue distribution as $\rho_{GT}(\lambda)$, in this case, is justified in the appendix.

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4.2 Other confining vacua

For any arbitrary superpotential (1.2) we always have a vacuum with classically unbroken gauge symmetry. For simplicity, we will take $W(\Phi)$ to be even.

In the matrix model

$$Z = \int [D\Phi] \exp\left(-\frac{N}{\mu} TrW(\Phi)\right), \tag{4.9}$$

the confining vacuum corresponds to expanding around the classical vacuum where all the eigenvalues are at the origin. The corresponding matrix eigenvalue density $\rho_{MM}(\lambda)$ therefore has only one cut and takes the form (see for e.g. [43])

$$\rho_{MM}(\lambda) = \frac{1}{2\pi} P_{n-1}(\lambda)\sqrt{4\Lambda^2 - \lambda^2}, \qquad \lambda \in [-2\Lambda, 2\Lambda]. \tag{4.10}$$

The actual polynomial $P_{n-1}(x)$ depends on the potential $W(\Phi)$ [43]. This can be thought of as a special case of the riemann surface (2.5) in which $n-1$ pairs of the branchpoints (a_i^-, a_i^+) have coalesced. In other words

$$F_{2n}(x) \propto P_{n-1}^2(x)(x^2 - 4\Lambda^2). \tag{4.11}$$

The corresponding planar moments will depend very much on $P_{n-1}(\lambda)$ and hence the details of the potential.

However, as we explained in the introduction, we expect the gauge theory answers for this vacuum to be independent of the detailed form of the potential. We see this from our prescription in the following way. From (4.11) and (3.12) it follows that

$$\omega = \frac{R_{n-1}(x)dx}{cP_{n-1}(x)\sqrt{x^2 - 4\Lambda^2}}, \tag{4.12}$$

where c is a constant. We saw earlier that the only poles of ω are at infinity. This continues to be true when the riemann surface degenerates due to the coalescing branchpoints. (The period integrals (3.13) over the other $n-1$ branch cuts are zero in this case since all the eigenvalues are spread about the origin. After they coalesce, these vanishing period integrals around the erstwhile branchpoints points imply that they are regular points, not poles.) Therefore, in (4.12) the potential poles from the zeroes of P_{n-1} must cancel against the zeroes of R_{n-1}. In other words, for this vacuum,

$$R_{n-1}(x) = cP_{n-1}(x).$$

This implies that

$$\rho_{GT}(\lambda) = \frac{1}{\pi}\frac{1}{\sqrt{4\Lambda^2 - \lambda^2}}, \tag{4.13}$$

as for the gaussian. The moments in the gauge theory also continue to be given by (4.5). Thus our prescription enables us to extract the universal behaviour of the confining vacua out of the highly non-universal behaviour of the matrix model.

5. Final comments

We have made a start in this paper to precisely characterise the notion of a geometric master field at least for a class of $\mathcal{N}=1$ theories. While we gave a mathematically precise prescription in the matrix model geometry, it would be nice to have a physical explication of the relation between $\rho_{MM}(\lambda)$ and $\rho_{GT}(\lambda)$. In this context we would like to point out that using the matrix model prescription for extracting $\rho_{GT}(\lambda)$, we can compute the low energy superpotential very simply:

$$W_{low}(g_p,\Lambda)=\sum_{p=2}^{n+1}\frac{g_p}{p}\langle Tr\Phi^p\rangle_{GT}=N\oint_C\rho_{GT}(\lambda)W(\lambda)d\lambda\,, \tag{5.1}$$

where C is a contour encircling all the cuts. This is different from the Dijkgraaf-Vafa prescription (reviewed in section 2) for $W_{\rm eff}$ which after minimisation gives

$$W_{low}(g_p,\Lambda)=W_{\rm eff}(g_p,\langle S_i\rangle,\Lambda)=\sum_i N_i\frac{\partial F_0}{\partial S_i}\Big|_{\langle S_i\rangle}=N\sum_i\nu_i\int\frac{\partial}{\partial S_i}(\mu\rho_{MM}(\lambda))\Big|_{\langle S_i\rangle}W(\lambda)d\lambda\,. \tag{5.2}$$

Here we have used the definition of F_0 in (2.11); we also recall from (2.12) that $\mu=\sum_i S_i$. Note that the term in $W_{\rm eff}$ linear in S_i, does not contribute to W_{low} after minimisation as it cancels against a nonperturbative contribution to the free energy, from the volume of the unbroken gauge group. Comparing with (5.1) suggests that

$$\rho_{GT}(\lambda)=\sum_i\nu_i\frac{\partial}{\partial S_i}(\mu\rho_{MM}(\lambda))\Big|_{\langle S_i\rangle}\,. \tag{5.3}$$

In fact, after the first version of this paper appeared, a relation was presented in [31] between gauge theory vevs and those of the matrix model, which is equivalent to (5.3). The argument in [31] (attributed to Vafa), relies on deforming the matrix model action (2.11) by an infinitesimal perturbation $\frac{\epsilon}{k}Tr\Phi^k$ and looking at the effect on $W_{\rm eff}$.

We can also see from (5.3) that $\rho_{GT}(\lambda)$ takes the form (3.10). As in [32] we simply have to change variables while minimising $W_{\rm eff}$. We take the derivatives in (5.3) w.r.t. to the variables b_k instead of S_i

$$\rho_{GT}(\lambda)=\sum_{i,k}\nu_i\frac{\partial b_k}{\partial S_i}\frac{\partial}{\partial b_k}(\mu\rho_{MM}(\lambda))\,. \tag{5.4}$$

where we parametrise the polynomial f_{n-1} in (2.4) as

$$f_{n-1}(x)=\sum_{k=0}^{n-1}b_kx^k\,.$$

From (3.11),(2.4) and (2.12) we then see that

$$\rho_{GT}(\lambda)=i\sum_{i,k}\nu_i\frac{\lambda^k}{\sqrt{-F_{2n}(\lambda)}}\Big(\oint_{\alpha_i}\frac{x^k}{y}dx\Big)^{-1}, \tag{5.5}$$

and hence $\rho_{GT}(\lambda)$ in (5.3) is of the form (3.10). It would be nice if these different forms for $\rho_{GT}(\lambda)$ had a direct physical interpretation in the matrix model.

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Though the master field as a concept is intrinsic to the large-N limit, it is curious that in these $\mathcal{N}=1$ theories, many large-N results go over to finite N. There should be some systematic way of understanding this by thinking of the large-N limit as a classical limit. It will also be interesting to understand the special points in parameter space of these theories where the conventional large-N limit breaks down (see [23] for a recent discussion of possible double scaling limits).

The role of the Cuntz oscillators in section 4.1 is intriguing, but might well be an accident particular to the gaussian case. More insight into all the above questions will also probably be had by generalising to other $\mathcal{N}=1$ systems.

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A. Cuntz algebras and eigenvalue distributions

Here we will show that the Cuntz operator $\hat{M}=\Lambda(a+a^\dagger)$ has a distribution of eigenvalues

$$\rho_C(\lambda)=\frac{1}{\pi}\frac{1}{\sqrt{4\Lambda^2-\lambda^2}}\,. \tag{A.1}$$

We can construct finite $K\times K$ matrices a_K, $a_K^\dagger$ which approximate to the Cuntz oscillators $a, a^\dagger$.

$$(a_K)_{ij}=\delta_{i,j-1}\,; \qquad \left(a_K^\dagger\right)_{ij}=\delta_{i-1,j}\,, \qquad (i,j=0,\ldots,K-1)\,. \tag{A.2}$$

a_K (and its adjoint $a_K^\dagger$) are just shift matrices with nonzero entries just above (below) the diagonal. Note that these are different from the usual 't Hooft shift matrices which have a nonzero corner entry which makes them unitary. Then $a_K|0\rangle=0$ and also

$$a_Ka_K^\dagger=\mathbf{I}-|K-1\rangle\langle K-1|\,, \qquad a_K^\dagger a_K=\mathbf{I}-|0\rangle\langle 0|\,, \tag{A.3}$$

where $\mathbf{I}$ is the $K\times K$ identity matrix. Our matrix indices $0,\ldots,K-1$ reflect the notation for the kets which are just the canonical unit vectors in the K-dimensional vector space on which these matrices act. These approximate, as $K\to\infty$, the Cuntz algebra $aa^\dagger=1$, $a^\dagger a=1-|0\rangle\langle 0|$ together with $a|0\rangle=0$.

It is not difficult to verify that $\hat{M}_K=\Lambda(a_K+a_K^\dagger)$ has eigenvalues

$$\lambda_k=2\Lambda\cos\frac{\pi k}{K+1}\,, \qquad (k=1,\ldots,K)\,. \tag{A.4}$$

In the large K limit this leads to the distribution (A.1).

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